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ABSTRACT

The author presents an organizational model which exhibits relationships among preservcie teachers' skills in the use of interpretation of logical connectives in mathematical contexts. The model also provides direction to teahcers of courses on mathematics and mathematics methods who seek to acquaint their students with those notions of logic recommended by CUPM. The model treats the translation from mathematical statements to logical statements, and conversely. An empirical test has yielded considerable support for the model. After a brief description of the model and this test, the author discusses implications for instruction, especially as they are related to statements and context of mathematical problems. (MN)

A Combinational Model for the Interpretation and Use of Propositions in the Context of Elementary Mathematics *

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Over a period of several years mathematics educators have been interested in the ability of teachers of elementary and secondary school mathematics to use propositional logic. Evidence of this interest abounds in several forms: many programs for preservice training of mathematics teachers require course-work on elementary logic; a large proportion of elementary teachers' textbooks on mathematics and mathematics methods include chapters or units on logic. Logic is among the topics with which the Committee on the Undergraduate Program in Mathematics (1971) recommended elementary and secondary teachers of mathematics be conversant, and the National Council of Teachers of Mathematics, in its Guidelines for the Preparation of Teachers (1974) made a similar recommendation. Articles such as the Exner-Hilton debate (Exner 1971, Hilton 1971) have appeared in journals devoted to the teaching of mathematics, and research on teachers' use of logical connectives (Gregory 1972, Gregory and Osborne 1975) and their ability to make inferences (Easterday and Henry 1978, Eisenberg and McGinty 1974, Jansson 1975, Juraschek 1978) has been reported in journals devoted to research on mathematics education.

However, there is ample evidence in many of the same sources that while logic might be judged important it is not considered "all that important" by mathematics educators. Although textbooks or programs may include chapters or courses on logic, these units are generally integrated poorly, if at all, with the remainder of the text materials; indeed textbooks on mathematics for

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elementary teachers frequently include logic as the last chapter, thus clearly demonstrating the independence of the preceding material from logic. Eisenberg and McGinty (1975) have examined the contents of textbooks for elementary teachers to determine the coverage of logic and concluded that it was generally inadequate. The writer has examined one popular mathematics textbook (Kelley and Richert 1970), coding all uses of logical connectives and concluded that except for a few difficult arguments (e.g., the Fundamental Theorem of Arithmetic) understanding of disjunctive, conditional, or biconditional connectives or constructions is unnecessary to an understanding of the text content.

This finding is not entirely inconsistent with the statement of the Committee on the Undergraduate Program in Mathematics (1971):

> ... the elementary notions of logic such as logical connectives, negation, and the quantifiers should be treated explicitly only after attention has been called informally to their uses in other mathematical contexts. Indirect proofs and the use of counterexamples arise naturally and should be stressed when the structure of the number systems is examined. However, in the final stages of an elementary teacher's training it is useful to return to logic in a more explicit way.....

However the ability of preservice elementary teachers to cope with indirect proof and the use of counterexamples in the absence of prior training on the use of propositional language and logic has been questioned, at least informally, by many teacher educators charged with teaching the structure of the number systems to preservice teachers.

Perhaps it is this nagging question which has led researchers such as Easterday and Henry (1978), Eisenberg and McGinty (1974), Jansson (1975) and Juraschek (1978) to examine performance of preservice elementary teachers on tests of inference. The results of these studies have revealed poor performance on items involving disjunctive, conditional, and biconditional statements. However, it should be noted that the test items used by these researchers did not refer to mathematical concepts or arguments, but rather to familiar objects (cars, bikes, sisters, etc.). There would appear to be no reason to believe that training such as that recommended by CUPM should generalize to the colloquial contexts of these items; nor, conversely, is it obvious that performance on these items should transfer to mathematical contexts. Thus the importance of these studies to the concerns of CUPM and others is unclear. Moreover, even if the relevance of these studies is established, they provide little direction to the teacher educator.

Purpose, Assumptions, and Definitions

The purpose of the analyses and studies described below is to construct an organizational model, analogous to a learning hierarchy, which at once exhibits relationships among preservice teachers' skills in the use and interpretation of logical connectives in mathematical contexts, and provides direction to teachers of courses on mathematics and mathematics methods who seek to acquaint their students with those notions of logic recommended by CUPM.

A major assumption underlying this effort asserts that one of the principal factors determining the ability of individuals to use and interpret propositional language in a manner consistent with deductive logic, as well as the appropriateness of that usage, is the context in which the language is presented or requested. The context has at least three aspects: (1) the concepts or phenomena with which the statements deal, (2) the availability of additional information concerning the concepts or phenomena, and (3) the nature of the interpretive task. What is "natural" or linguistically appropriate in some contexts might be "formal" or even artificial, as well as linguistically inappropriate, in other contexts, and vice-versa. The fundamental relationships between logic and mathematics imply an *a priori* importance to logic in mathematical contexts;

moreover, these relationships lend a "natural" character to propositional statements about mathematical concepts.

Parts of this assumption are bolstered by examination of the research literature on the use of logic. Although the effects of mathematical vs. nonmathematical contexts have not yet been sufficiently studied several differences are apparent when one compares the results of studies in different contexts. The most striking of these concerns the disjunction. Several researchers (e.g., Jurashek 1978) using tests concerning nonmathematical concepts or objects have found strong evidence that subjects interpret the word "or" exclusively. In two studies using mathematical concepts (set membership, odd and even numbers), on the other hand, Damarin (1977a, 1977b) found that approximately a third of the subjects interpreted "or" inclusively, while half of the subjects treated "or" as if it were "and;" no subject consistently treated "or" as an exclusive disjunction.

A second assumption is that the importance of elementary teachers' understanding of logical connectives is complex. Not only should teachers be able to draw valid inferences from clearly stated pairs (or strings) of statements; they should also be able to translate such statements into other forms such as equations, inequalities, and listings of truth sets, and conversely. Indeed, some of these translational abilities are necessary to the full understanding of the statements, and, therefore, to the consistent drawing of appropriate inferences. The forms in which statements and their translations are presented will be called "presentation modes."

At least three modes of presenting information concerning mathematical relations among variables as they range over dichotomously partitioned sets (e.g. odd and even integers, zero and nonzero counting numbers, etc.) can be identified. These three presentation modes are:

- Logical statements: Simple statements (p,q) or compound statements in the forms "p and q," "p or q," "if p then q," and "p if and only if q;" neither p nor q refer directly to mathematical operations.
- Mathematical statements: Simple statements whose subjects are (the results of) mathematical operations, or statements (e.g. equations) expressing mathematical relations between algebraic expressions.

- Sortings of the set of possibilities: Partitions of the set of possible "values" (with respect to the dichotomous partition) of the variables into "true" and "false" sets.

Examples of these presentation modes for situations involving two variables and the partition of the integers into classes of odd and even numbers are presented in table 1.

Table 1
Presentation of Equivalent information in 3 modes

Mathematical Statement	Logical Statement	Sorting of the replacement set
M + N is odd	M is even if and only if N is odd	Truth set: [(M odd, N even), (M even, N odd)] Complement: [(M odd, N odd), (M even, N even)]
M x N is even	M is even or N is even (or) If M is odd, then N is even	Truth set: [(M odd, N even), (M even, N odd), (M even, N even)] Complement: [(M odd, N odd)]
M x N is odd	M is odd and N is odd	Truth set: [(M odd, N odd)]

Any piece of information which can be presented in one of these modes can also be stated in either of the others; translation from one mode to another is frequently a critical component of indirect proof and of the identification of counterexamples. There are six translation tasks which can be performed among the three modes; these are identified and named in table 2.

See next page for Table 2

Table 2
Descriptions of six translation tasks

Task Name	Translation Task
L	Given a logical statement, sort the set of possibilities by indicating for which possibilities the statement is true.
M	Given a mathematical statement, sort the set of possibilities by indicating for which possibilities the statement is true.
LM	Given a logical statement select an equivalent mathematical statement.
ML	Given a mathematical statement select an equivalent logical statement.
L'	Given a sorting of the set of possibilities select a logical statement describing the true set.
M'	Given a sorting of the set of possibilities select a mathematical statement describing the true set.

Questions concerning the relationships among these tasks and between these tasks and the inference process can be posed at many levels. Are these translations constituent parts of the inference process? Do some translations occur as parts of others? For example, does performance of translation LM consist of performing first L and then M'? Can individuals who can perform one translation also perform its reverse? Can translation LM be performed by persons who cannot perform translation L? Numerous other questions might be addressed.

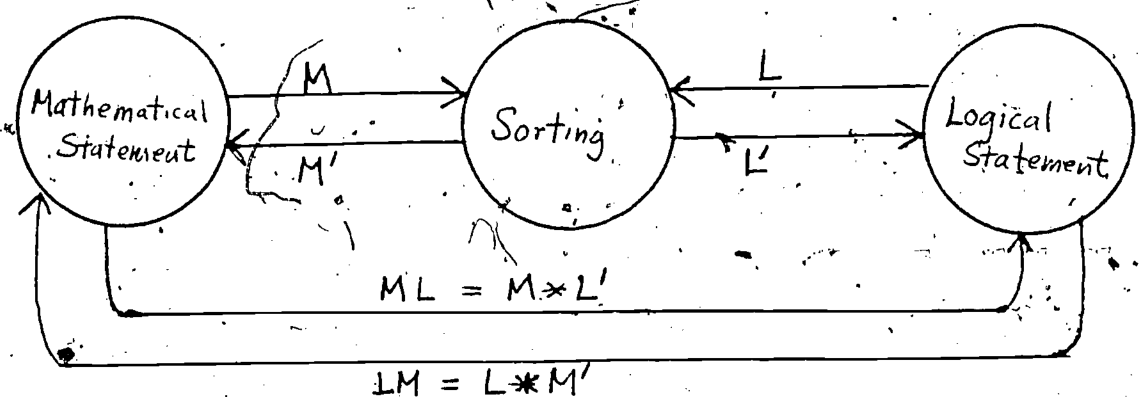
It is not the purpose here to address the question of whether the translation processes outlined above are actually involved in the cognitive processing of statements and the extraction of inferences from them. Rather the point of view adopted is that individuals draw inferences on the bases of their understandings of the meanings of statements, and that the depth of understanding is related to the ability to translate the statement to equivalent statements in the same mode or in other modes. Moreover, systematic errors in inference can be explained by misunderstandings which are revealed in systematic translation errors. The model described below is therefore posed, not as a potential model for the actual cognitive processing of statements, but rather as a model of the organization of skills related to the understanding of, translation of, and inference from statements about mathematical concepts and variables.

The Model

The understanding of mathematical arguments, proof, and counterexample is dependent upon the understanding of mathematical statements, understanding of logical statements, and the ability to make translations and comparisons back and forth between logical and mathematical modes.

The understanding of statements in each mode has two aspects: interpretive understanding which is operationally defined as the ability to partition the replacement set into the truth set and its complement, and constructive understanding which is the ability to use the statement to describe the appropriate sorting for the replacement set. The ability to make translation L or M listed in table 2 is indicative of interpretive understanding of the given statement, while the ability to make translation L' or M' reveals constructive understanding.

Translation from a mathematical statement to a logical statement requires interpretation of the mathematical statement and construction of the logical statement from the interpretation. An analogous pair of operations is required for translation from a logical to a mathematical statement. Using the operation names assigned in table 2, ML, translation from a mathematical to an equivalent logical statement is modelled as the composite of skills M and L! Similarly LM is modelled as the composite of skills L and M'. Figure 1 below illustrates the model. In the absense of research evidence establishing relationships between or among L, L', M, and M', these skills are considered to be independent of each other. On the basis of the *a priori* analysis above, LM is considered as L*M' and ML as M*L'.



Empirical Studies: Support for the Model

A number of investigations related to this model have been conducted using populations of preservice elementary teachers (Damarin 1976, 1977a, 1977b, 1978, to appear). These studies have shown that L and L', interpretive and constructive understanding of logical statements about mathematical variables are distinct abilities with constructive understanding being more prevalent among the population than interpretive understanding (1976, to appear). Translation from mathematical to logical statement is correspondingly easier for the population than the converse translations. Interpretive and constructive understanding of the mathematical statements tested were correlated in this population.

The hypothesis that the ability to translate from a mathematical to a logical statement is dependent upon the interpretive understanding of the mathematical statement and the constructive understanding of the logical statement was supported. However the difficulty of both the test of interpretative understanding of logical statements and the test for translation from logical to mathematical statements precluded drawing any conclusion except that these skills are rare in the population.

Empirical Studies: Findings related to interpretive and constructive understanding of logical statements. Two studies (1977a, 1977b) revealed that preservice elementary teachers interpret both conditional and biconditional statements about mathematical concepts in the same way they interpret conjunctions, that is, the only member of the replacement set assigned to the truth set is the element satisfying both simple statements. The same studies showed that slightly less than one-third of the preservice elementary teachers tested interpreted the connective "or" as an inclusive disjunction, and more than forty percent as a conjunction (the remainder were inconsistent in their responses). Response patterns on tests of constructive understanding were observed to be similar but less well-defined (1978).

Because the evidence concerning interpretation of "or" stands in stark contrast to the findings of other researchers (Eisenberg and McGinty 1974, Jansson 1975, Juraschek 1978) who used inference tests in non-mathematical contexts, an attempt was made to determine whether the context of statements determined the interpretive understanding of the word "or." (Damarin to appear). Inference items and interpretive understanding items were constructed in mathematical, technical-scientific, and familiar content areas yielding six tests. When tests were randomly ordered and administered to preservice elementary teachers it was found that these subjects were more likely to make inclusive disjunctive interpretations or conjunctive interpretations of "or" on both interpretive and inference tests when statements dealt with mathematics than when statements were drawn from the other contexts. An order effect was also apparent in the data, however; after some inconsistency on the first several items many subjects tended to settle on a single interpretation of "or" and use it on the remaining items regardless of context.